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VI.

ON THE MATRICAL EQUATION $\phi\Omega = \Omega\phi$.

By Henry Taber, Clark University.

Presented by Prof. W. E. Story, May 26, 1891.

For a given matrix, Ω, the most general matrix ϕ satisfying the equation $\phi\,\Omega = \Omega\,\phi$ may be found by the consideration of the canonical form of the matrix Ω. If the distinct latent roots of Ω are g_1, an m-tuple latent root, g_2, an n-tuple latent root, etc., the canonical form of Ω is $\omega\,\theta\,\omega^{-1}$, where

$$\theta = \begin{pmatrix} \boxed{\theta_1} & & & \\ & \cdot\ \boxed{\theta_2} & & \\ & & \cdot & \\ & & & \cdot\ \cdot\ \cdot \end{pmatrix}$$

in which all the constituents are zero except those in the square arrays θ_1, θ_2, etc., which correspond, respectively, to the latent roots g_1, g_2, etc., and are severally of order equal to the multiplicity of the latent root to which they correspond; and if the characteristics of the latent root g_1 are $(m\,;\ p,\ q,\ r,\ \ldots.\ s,\ t)$, then

$$\begin{array}{cc|c|c|c|c|c|c|}
 & & \overbrace{\qquad}^{p} & \overbrace{\qquad}^{q} & \overbrace{\qquad}^{r} & & \overbrace{\qquad}^{s} & \overbrace{\qquad}^{t} \\
\theta_1 = & p\left\{\begin{matrix}\\ \\ \\ \\ \end{matrix}\right. & \begin{matrix} g_1 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{matrix} & \begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{matrix} & & & & \\
 & q\left\{\begin{matrix}\\ \\ \\ \\ \end{matrix}\right. & & \begin{matrix} g_1 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & g_1 \end{matrix} & \begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{matrix} & & & \\
 & r\left\{\begin{matrix}\\ \\ \\ \\ \end{matrix}\right. & & & \begin{matrix} g_1 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & g_1 \end{matrix} & & & \\
 & & & & & \ddots & & \\
 & s\left\{\begin{matrix}\\ \\ \\ \\ \end{matrix}\right. & & & & & \begin{matrix} g_1 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & g_1 \end{matrix} & \begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{matrix} \\
 & t\left\{\begin{matrix}\\ \\ \\ \\ \end{matrix}\right. & & & & & & \begin{matrix} g_1 & 0 & \dots & 0 \\ 0 & g_1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & g_1 \end{matrix}
\end{array}$$

where the letters on the left and above denote the number of rows and columns in the respective squares and rectangles below and to the right. Similarly with respect to the square arrays θ_2, etc., corresponding to the other latent roots.* If, now, ϕ is a matrix commutative with Ω, the most general expression for ϕ is $\phi = \omega\, \eta\, \omega^{-1}$, where

$$\eta = \left(\begin{array}{cccc} \boxed{\;\eta_1\;} & & & \\ & \boxed{\;\eta_2\;} & & \\ & & \ddots & \\ & & & \end{array}\right)$$

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* This representation of a matrix, what I term its canonical or standard form, was, I believe, first given explicitly by Weyr, Comptes Rendus, Vol. C.; it was, however, given substantially by Buchheim, earlier, in Proc. Lond. Math. Soc., Vol. XVI.

in which all the constituents are zero except those in the squares along the principal diagonal, and

$$\eta_1 = \left|\begin{array}{c|c|c|c}
\begin{array}{l} a_{11}\ a_{12}\dots a_{1p}\\ a_{21}\ a_{22}\dots a_{2p}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ a_{p1}\ a_{p2}\dots a_{pp}\end{array} & \begin{array}{l} b_{11}\ b_{12}\dots b_{1q}\\ b_{21}\ b_{22}\dots b_{2q}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ b_{p1}\ b_{p2}\dots b_{pq}\end{array} & \begin{array}{l} c_{11}\ c_{12}\dots c_{1r}\\ c_{21}\ c_{22}\dots c_{2r}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ c_{p1}\ c_{p2}\dots c_{pr}\end{array} & \text{etc.}\\
\cline{1-3}
 & \begin{array}{l} a_{11}\ a_{12}\dots a_{1q}\\ a_{21}\ a_{22}\dots a_{2q}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ a_{q1}\ a_{q2}\dots a_{qq}\end{array} & \begin{array}{l} b_{11}\ b_{12}\dots b_{1r}\\ b_{21}\ b_{22}\dots b_{2r}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ b_{q1}\ b_{q2}\dots b_{qr}\end{array} & \text{etc.}\\
\cline{2-3}
 & & \begin{array}{l} a_{11}\ a_{12}\dots a_{1r}\\ a_{21}\ a_{22}\dots a_{2r}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ a_{r1}\ a_{r2}\dots a_{rr}\end{array} & \text{etc.}\\
\cline{3-3}
 & & & \text{etc.}
\end{array}\right|$$

the *a*'s, *b*'s, *c*'s, etc. being arbitrary; the mode of filling up the remaining squares along the principal diagonal and the rectangles above the principal diagonal is obvious. A similar expression obtains for η_2, etc.

WORCESTER, May 26, 1891.